

Braneworld teleparallel gravity

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We study the gravity in the context of a braneworld teleparallel scenario. The geometrical setup is assumed to be Randall-Sundrum II model where a single positive tension brane is embedded in an infinite AdS bulk. We derive the equivalent of Gauss-Codacci equations in teleparallel gravity and junction conditions in this setup. Using these results we derive the induced teleparallel field equations on the brane. We show that compared to general relativity, the induced field equations in teleparallel gravity contain two extra terms arising from the extra degrees of freedom in the teleparallel Lagrangian. The term carrying the effects of the bulk to the brane is also calculated and its implications are discussed.

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I. INTRODUCTION

The theory of general relativity (GR) first introduced by Einstein in 1916, is one of the great achievements of theoretical physics. The remarkable agreement of GR with experimental data, at least in the solar system scales, turns it into one of the most successful theories ever developed. However there are some dark spots that it is not able to answer. For example, the theory of general relativity effectively predicts its own demise with the prediction of singularities. Also the problem of combining gravity with quantum theory and the issue of unification of fundamental forces, make modifying GR a more plausible task. It is obvious that in extremely high energy levels (Planck scale) the theory of GR breaks down, hence the idea of modifying gravity came to existence. One of these ideas is higher dimensional world theories which originated by the work of Kaluza and Klein [1]. They considered the extra dimension in their attempt to unify electromagnetism with gravity. In 1998, Arkani-Hamed, Dimopoulos and Dvali (ADD) showed that the extra dimension can be in order of 0.1mm [2]. The extra dimensions of the ADD scenario are flat and the model postulates that the Standard Model fields are confined to a 4-dimensional brane, with only gravity propagating in the bulk. In the two Randall-Sundrum (RS) models proposed shortly afterwards, the extra dimension is not flat and the bulk geometry is curved. This leads the brane(s) to have tension so the brane(s) and the bulk can dynamically interact with each other. Similar to the ADD scenario, in the first Randall-Sundrum model (RS I) the extra dimension is still compact [3]. The RS I model consists of two branes of tensions λ_1 and λ_2 bounding a slice of anti-de Sitter (AdS) space. The main aim of the first RS model was to solve the Hierarchy problem. The inter-brane separation in this model is characterized by an extra degree of freedom called Radion. To get the de-

sired result of the theory the Radion should be stabilized which brings up some complexities [4]. Moreover, the model leads to unconventional and unacceptable cosmology in the weak-field limit on the brane. In the second RS model (RS II), the extra dimension is infinite in size and the bulk curvature causes the gravity to be localized on the brane [5]. The RS II model consists of a single, positive tension brane in an infinite bulk. The model can be thought of as a RS I model when the negative tension brane is moved to infinity. This model does not address the hierarchy problem, but has some remarkable gravitational and cosmological implications. The elegance and geometrical simplicity of this model had a huge impact on studies of extra dimensions and led to vast literature researching various aspects of it in gravity [6] and cosmology [7]. This is the model we use as our geometric framework in this paper. We follow the procedure introduced by Shirumizu, Maeda and Sasaki in 2000 [8]. They obtained the 4-Dimensional field equations on the brane by a geometrical approach projecting the 5-D quantities to the brane.

On the other hand, in 1928, along with general relativity Einstein presented another form of theory of gravity called teleparallel gravity [9]. In this theory, a set of four tetrad (or vierbein) fields form the (pseudo)-orthogonal bases for the tangent space at each point of spacetime and the torsion instead of curvature describes gravitational interactions. Tetrads are the dynamical variables and play a similar role to the metric tensor field in general relativity. Teleparallel gravity also uses the curvature-free Weitzenböck connection instead of Levi-Civita connection of general relativity to define covariant derivatives [10].

Since the first introduction, some innovative works developed the theory and it has been shown that teleparallel Lagrangian density only differs with Ricci scalar by a total divergence [11, 12]. This shows that general relativity and teleparallel gravity are dynamically equivalent theories where the difference arises only in boundary terms. However, there are some fundamental conceptual differences between teleparallel theory and general relativity. According to general relativity, gravity curves

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the spacetime and shapes the geometry. In teleparallel theory however torsion does not shape the geometry but instead acts as a force. This means that there are no geodesics equations in teleparallel gravity but there are force equations much like the Lorentz force in electrodynamics. In teleparallel gravity one would consider the nontrivial characteristics of spacetime in the (pseudo) orthogonal bases of the tangent space (tetrad) and the spacetime is considered flat. The relation between manifold and Minkowski metrics is

$$g_{\mu\nu} = \eta_{ij} e_\mu^i e_\nu^j \quad (1)$$

The Greek indices referred to coordinate basis of the manifold and Latin indices referred to basis of the tangent space. Both indices run from 0 to the dimension of the spacetime. Only the spin connection A_{bc}^a acts on tangent space indices and the Weitzenböck connection $\Gamma_{\mu\nu}^\rho$ only acts on spacetime indices. The spin connection in teleparallel gravity is only related to Weitzenböck connection like general relativity which is only related to Livi-Civita connection. To satisfy the absolute parallelism condition in teleparallel gravity, the spin connection is supposed to be vanished. This leads to lack of spinor fields which guarantees the equivalence of the two theories. One can define the curvature and torsion with respect to spin connection and vierbeins [12] which in teleparallel gravity leads to zero curvature and non-zero torsion as

$$T_{\mu\nu}^\rho \equiv e_i^\rho (\partial_\mu e_\nu^i - \partial_\nu e_\mu^i) \quad (2)$$

In attempts to obtain the accelerated expansion of the universe the extended version of the theory has been presented. In $f(T)$ gravity, in analogy with $f(R)$ the higher order terms of T is considered in the Lagrangian of teleparallel action. In the case of $f(T) = T$ the Lagrangian can be assumed to be locally Lorentz invariant. Nevertheless, for $f(T) \neq T$ the theory is not locally Lorentz invariant anymore. This property would bring more degrees of freedom to the theory in comparison with $f(R)$ gravity[13]. For more information about the theory and its extended versions see [14] and references therein. Notwithstanding the equivalence of the two theories, in the situations where boundary and induced terms are present, teleparallel gravity would not be equal to general relativity anymore. The reason for this is that for a given “induced” metric, there exist an infinite number of “induced” vierbeins that give the same metric and each of these vierbeins results in different field equations. Such boundary terms exist in brane world theories which brings up the questions about the interpretation of gravity in higher dimensional teleparallel scenarios. We wish to answer some of these questions in this paper, using the RS II model as our braneworld geometrical setup.

The structure of the paper is as follows: In section II we review basic definitions of teleparallel gravity and set out the notation for the rest of the paper. In section III, we derive the equivalent of Gauss-Codacci equations in teleparallel gravity using our definition of projection vierbein. In section IV, the junction conditions in our

model is obtained. In section V, using the results of previous sections and following the procedure of reference [8], we derive the induced teleparallel field equations on the brane and finally in section VI conclusion and some discussions are presented.

II. NOTATION AND DEFINITIONS

In brane teleparallel gravity we define five tetrad fields $e_i(x^\mu)$ which form the basis of tangent space at each point of the manifold with spacetime coordinates x^μ : $e_i \cdot e_j = \eta_{ij}$. Latin indices labels the tangent space coordinates while Greek indices label spacetime coordinates. Both set of indices take the values 0, 1, 2, 3, 5 to agree with conventional notation. $e_i(x^\mu)$ is a vector in the tangent space and can be expressed in terms of its components in the coordinates space as e_i^μ which is called the vierbein. Vierbeins are also vectors in spacetime. The relation between the Minkowski and the spacetime metrics is as mentioned in equation (1). The inverse vierbein is defined by the relation $e_i^\mu e_\mu^j = \delta_i^j$. The vierbein is also a matrix that transforms between the tetrad frame and the coordinate frame.

The connection in teleparallel theory, the Weitzenböck connection, is defined as

$$\Gamma_{\mu\nu}^\rho = e_i^\rho \partial_\nu e_\mu^i \quad (3)$$

By this definition, the torsion tensor and its permutations are [11]

$$T_{\mu\nu}^\rho \equiv e_i^\rho (\partial_\mu e_\nu^i - \partial_\nu e_\mu^i) \quad (4)$$

$$K^{\mu\nu}{}_\rho = -\frac{1}{2}(T^{\mu\nu}{}_\rho - T^{\nu\mu}{}_\rho - T_\rho{}^{\mu\nu}) \quad (5)$$

$$S_\rho{}^{\mu\nu} = \frac{1}{2}(K^{\mu\nu}{}_\rho + \delta_\rho^\mu T^{\alpha\nu}{}_\alpha - \delta_\rho^\nu T^{\alpha\mu}{}_\alpha), \quad (6)$$

where $K^{\mu\nu}{}_\rho$ is the contortion tensor which is the difference between Levi-Civita and the Weitzenböck connections and $S_\rho{}^{\mu\nu}$ is called the superpotential. It should be noted that the superpotential tensor carries all the extra degrees of freedom in teleparallel gravity. In fact, due to (anti-)symmetrical properties of the Weitzenböck connection, compared to general relativity, there are more possible contractions that can be used to define the Lagrangian of the theory. In correspondence with Ricci scalar we define a torsion scalar as

$$T = S_\rho{}^{\mu\nu} T_{\mu\nu}^\rho \quad (7)$$

so the gravitational action is

$$I = \frac{1}{16\pi G} \int d^5x |e| T \quad (8)$$

where $|e|$ is the determinant of the vierbein e_μ^a which is equal to $\sqrt{-g}$. Variation of the above action with respect

to the vierbeins will give the (5-dimensional) teleparallel field equations

$$e^{-1}\partial_\mu(ee_i^\rho S_\rho^{\mu\nu}) - e_i^\lambda T^\rho_{\mu\lambda} S_\rho^{\nu\mu} + \frac{1}{4}e_i^\nu T = 4\pi G e_i^\rho \Xi_\rho^\nu \quad (9)$$

where Ξ_ρ^ν is the 5-dimensional matter energy-momentum tensor. Since in the Randall-sundrum II setup, the bulk is assumed to be empty except for a cosmological constant and the matter is assumed to be confined to the brane, we can decompose the 5-dimensional stress-energy tensor as

$$\Xi_{\mu\nu} = -\Lambda_5 g_{\mu\nu} + \delta(y) \frac{8\pi}{M_5^3} [-\lambda g_{\mu\nu} + \tau_{\mu\nu}] \quad (10)$$

where Λ_5 is the 5-dimensional cosmological constant, λ is the brane tension and $\tau_{\mu\nu}$ is the matter stress-energy tensor of the brane which is assumed to be located at $y = 0$.

III. INDUCED QUANTITIES ON THE BRANE: GAUSS-CODACCI EQUATIONS

Randall-Sundrum II geometry consists of a 4-dimensional hypersurface or brane embedded in a 5-dimensional AdS bulk. In order to derive the induced teleparallel field equations on the brane, it is necessary to know how the 5-dimensional quantities can be projected into the 4-dimensional hypersurface (brane). The projection tensor on the brane, $q_{\mu\nu}$ is defined as

$$q_{\mu\nu} = g_{\mu\nu} - n_\mu n_\nu \quad (11)$$

where n_μ is the unit normal vector on the brane. With this definition we can also define a projection vierbein

$$h_i^\mu = e_i^\mu - n^\mu n_i. \quad (12)$$

One should note that the projection vierbein defined here is not unique. One can define other vierbeins to give the same induced metric (11). h_i^μ acts as a projection operator on the hypersurface as one can easily check. Any vector v^i can be decomposed into parts tangent and perpendicular to the hypersurface $v^i = v_\parallel^i + v_\perp n^i$. Acting the projection vierbein on this, we get

$$h_i^\mu v^i = (e_i^\mu - n^\mu n_i)(v_\parallel^i + v_\perp n^i) = v_\parallel^\mu \quad (13)$$

so the projection vierbein projects the vector to the hypersurface and also changes indices from coordinate to tangent and vice-versa. Using the above equation we can define a covariant derivative operator on the brane, D_μ by projecting all the indices in the 5-dimensional covariant derivative using h_i^μ

$$D_\mu T^{\alpha_1 \dots \alpha_n}_{\beta_1 \dots \beta_m} = h_{i_1}^{\alpha_1} \dots h_{i_n}^{\alpha_n} h_{j_1}^{\beta_1} \dots h_{j_m}^{\beta_m} h_\mu^k \nabla_k T^{i_1 \dots i_n}_{j_1 \dots j_m} \quad (14)$$

In order to derive the equivalent of the Gauss-Codacci equations in teleparallel gravity, we start with the following general equation. For any vector X^a we have

$$\nabla_d \nabla_c X^a - \nabla_c \nabla_d X^a = R^a_{bcd} X^b + T^e_{cd} \nabla_e X^a \quad (15)$$

In teleparallel gravity, where the connection is Weitzenböck, the Riemann tensor identically vanishes, so we have

$$\nabla_d \nabla_c X^a - \nabla_c \nabla_d X^a = T^e_{cd} \nabla_e X^a \quad (16)$$

where ∇ is Weitzenböck covariant derivative here. Using the above equation we can write

$$D_\mu D_\nu X_\rho - D_\nu D_\mu X_\rho = {}^{(N-1)}T^\alpha_{\mu\nu} D_\alpha X_\rho \quad (17)$$

where N is the dimension of the spacetime (5 in Randall-Sundrum setup) and D_μ is the covariant derivative on the brane. Substituting (14) in equation (17) and also considering the fact that Weitzenböck covariant derivative of the tetrad vanishes, we arrive at the teleparallel equivalent of the Gauss equation after some straightforward algebra

$${}^{(N-1)}T^\rho_{\mu\nu} = h_k^\rho h_\nu^j h_\mu^i {}^{(N)}T^k_{ij} \quad (18)$$

The interesting point is that unlike general relativity where the extrinsic curvature explicitly enters the equation, the “extrinsic torsion” is not present in the R.H.S of the Gauss equation in teleparallel gravity. The reason lies in the form of equation (16). In general relativity the arbitrary vector appears in the R.H.S but in teleparallel gravity covariant derivative of the vector appears. Substituting 4-dimensional covariant derivative with 5-dimensional ones results in term involving the extrinsic torsion to drop out of the final result.

Starting with (18), we can now project all the terms in the L.H.S side of the teleparallel field equation on the brane (9). The results are

$${}^{(N-1)}S_\rho^{\mu\nu} = {}^{(N)}S_\rho^{\mu\nu} +$$

$$\left[n^\mu n_\rho n_i n^k e_j^\nu + n^k n_\rho n^\nu n_j e_i^\mu + \right.$$

$$n^\mu n_i n^\nu n_j e_\rho^k - n^k e_i^\mu n_\rho e_j^\nu - n^\mu n_\rho e_j^k e_i^\nu$$

$$\left. - n^\mu n_\rho n_i n^k n^\nu n_j \right] {}^{(N)}S_k^{ij} \quad (19)$$

$${}^{(N-1)}T^\rho_{\mu\lambda} {}^{(N-1)}S_\rho^{\nu\mu} = {}^{(N)}T^\rho_{\mu\lambda} {}^{(N)}S_\rho^{\nu\mu} +$$

$$\left[n^k n_\lambda n^\nu n_j - e_\lambda^k n^\nu n_j - e_\nu^j n^k n_\lambda \right] {}^{(N)}T^m_{ik} {}^{(N)}S_m^{ji} \quad (20)$$

and

$${}^{(N-1)}T = {}^{(N)}T \quad (21)$$

The left hand side of the teleparallel field equation which we denote by F_l^ν , can now be constructed by means of these relations

$$F_l^\nu = e^{-1}\partial_\mu(ee_l^\rho S_\rho^{\mu\nu}) - e_l^\lambda T^\rho_{\mu\lambda} S_\rho^{\nu\mu} - \frac{1}{4}e_l^\nu T \quad (22)$$

Combining (18),(19),(20) and (21) we have

$$\begin{aligned}
{}^{(4)}F_l^\nu = & \left[e^{-1} \partial_\mu (e e_l^\rho {}^{(5)}S_\rho^{\mu\nu}) - e_l^\lambda {}^{(5)}T_{\mu\lambda}^\rho {}^{(5)}S_\rho^{\nu\mu} \right. \\
& + \left. \frac{1}{4} e_l^\rho {}^{(5)}T \right] + \partial_\mu (n^\rho n_l {}^{(5)}S_\rho^{\mu\nu}) \\
& - n^\lambda n_l {}^{(5)}T_{\mu\lambda}^\rho {}^{(5)}S_\rho^{\nu\mu} \\
& + \frac{1}{4} n^\nu n_l {}^{(5)}T + h_l^\rho \partial_\mu \left[A_{\rho ij}^{\mu k\nu} {}^{(5)}S_k^{ij} \right] \\
& + A_{\rho ij}^{\mu k\nu} {}^{(5)}S_k^{ij} \partial_\mu (h h_l^\rho) \\
& - h_l^\lambda B_{\lambda j}^{k\nu} {}^{(5)}T_{ik}^m {}^{(5)}S_m^{ji}
\end{aligned} \quad (23)$$

where we have defined

$$\begin{aligned}
A_{\rho ij}^{\mu k\nu} = & n^\mu n_\rho n_i n^k e_j^\nu + n^k n_\rho n^\nu n_j e_i^\mu \\
& + n^\mu n_i n^\nu n_j e_\rho^k - n^k e_i^\mu n_\rho e_j^\nu \\
& - n^\mu n_\rho e_j^k e_i^\nu - n^\mu n_\rho n_i n^k n^\nu n_j
\end{aligned} \quad (24)$$

and

$$B_{\lambda j}^{k\nu} = n^k n_\lambda n^\nu n_j - e_\lambda^k n^\nu n_j - e_\nu^j n^k n_\lambda. \quad (25)$$

IV. JUNCTION CONDITIONS

In the geometrical setup of the Randall-Sundrum II model, the brane acts as a boundary hypersurface that connects the two “sides” of the bulk. In order to derive the induced field equations on the brane, it is necessary to know how the physical quantities change from one side of the brane to the other. This needs the so called junction conditions in order to deal with the discontinuities across the hypersurface. This problem has been discussed in the context of Einstein-Cartan manifolds where both curvature and torsion are present [15]. In this section we will derive the junction conditions in our braneworld teleparallel gravity setup.

The derivation is conducted in the Gaussian Normal Coordinates (GNC). In the context of general relativity, GNC is constructed as follows: On a given hypersurface Σ , the unique geodesic is constructed with tangent vector n^a through each point of Σ . Then each point in a neighborhood of Σ is labeled by the affine parameter y along the geodesic on which it lies and also by the arbitrary coordinates $(x_1; \dots; x_{n-1})$ of the point $p \in \Sigma$ from which the geodesic originated. Then $(x_1; \dots; x_{n-1}; y)$ defines the GNC system. This relation $n_a dx^a = dy$ then holds and the metric takes the form

$$ds^2 = g_{ab} dx^a dx^b = q_{\mu\nu} dx^\mu dx^\nu + dy^2 \quad (26)$$

The brane now can be chosen to be located at $y = 0$ without any loss of generality. A similar procedure can be applied in teleparallel gravity. Instead of using the geodesics to define the GNC coordinate system, we will use the teleparallel force equation which is equivalent to the geodesic equation of general relativity. Then the

tetrad in the Gaussian normal coordinates of teleparallel gravity takes the form

$$ds^2 = \eta_{ij} h_\mu^i h_\nu^j dx^\mu dx^\nu + dy^2 \quad (27)$$

This can be justified as follows. Similar to general relativity, with a metric in the form of (26), we can choose the unit normal vector n_μ to be purely in the direction of the extra dimension $n_\mu = (0, 0, 0, 0, 1)$. On the other hand, the counterpart of n_μ in the tangent space is denoted by n_i . At any point on the hypersurface (brane), the normal to the hypersurface is also normal to the tangent space at that point. This leads us to conclude that n_i should also be $n_i = (0, 0, 0, 0, 1)$, so for the 55 component of the vierbein we have $e_5^5 = 1$ and for the $\mu 5$ components we have $e_5^\mu = 0$.

In order to derive the junction conditions, we employ the language of distributions [16]. Heaviside distribution $\Theta(y)$ is defined as follows: it is equal to +1 if $y > 0$, 0 if $y < 0$ and indeterminate if $y = 0$. It has the following properties

$$\Theta^2(y) = \Theta(y) \quad , \quad \Theta(y)\Theta(-y) = 0 \quad , \quad \frac{d}{dy}\Theta(y) = \delta(y) \quad (28)$$

where $\delta(y)$ is the Dirac distribution. With this definitions, the tetrad field can be written in the following way

$$e_\mu^i = \Theta(y)e_\mu^{i(+)} + \Theta(-y)e_\mu^{i(-)} \quad (29)$$

where $e_\mu^{i(+)}$ and $e_\mu^{i(-)}$ denote the vierbeins in the $y > 0$ and $y < 0$ side of the brane respectively. The teleparallel connection, the Weitzenböck connection includes the first derivative of the vierbein. Differentiating (29) and denoting the jump of any tensorial property f across the brane by

$$[f] \equiv f^+ - f^- \quad (30)$$

we have

$$\partial_\rho e_\mu^i = \Theta(y)\partial_\rho e_\mu^{i(+)} + \Theta(-y)\partial_\rho e_\mu^{i(-)} + \delta(y)[e_\mu^i]n_\rho \quad (31)$$

and

$$\begin{aligned}
\Gamma_{\mu\nu}^\rho = & \Theta(y)\Gamma_{\mu\nu}^{\rho(+)} + \Theta(-y)\Gamma_{\mu\nu}^{\rho(-)} + \Theta(y)\delta(y)e_i^{\rho(+)}[e_\mu^i]n_\nu \\
& + \Theta(-y)\delta(y)e_i^{\rho(-)}[e_\mu^i]n_\nu.
\end{aligned} \quad (32)$$

If the last two terms remain, then the Weitzenböck connection will include $\Theta(y)\delta(y)$ terms which is not defined as a distribution; so the connection itself can not be written as a distribution. This leads to an ill-defined geometry. In order to avoid this we impose the condition

$$[e_\mu^i] = 0 \quad (33)$$

which if written in a coordinate independent way implies that

$$[h_\mu^i] = 0 \quad (34)$$

The equation above is the first junction condition. This means that the induced vierbein should be the same on both sides of the brane.

Now the connection can be written as

$$\Gamma^\rho_{\mu\nu} = \Theta(y)\Gamma^\rho_{\mu\nu}^{(+)} + \Theta(-y)\Gamma^\rho_{\mu\nu}^{(-)} \quad (35)$$

So the torsion tensor becomes

$$\begin{aligned} T^\rho_{\mu\nu} &\equiv \Gamma^\rho_{\mu\nu} - \Gamma^\rho_{\nu\mu} \\ &= \Theta(y)\Gamma^\rho_{\mu\nu}^{(+)} + \Theta(-y)\Gamma^\rho_{\mu\nu}^{(-)} \\ &\quad - \Theta(y)\Gamma^\rho_{\nu\mu}^{(+)} + \Theta(-y)\Gamma^\rho_{\nu\mu}^{(-)} \\ &= \Theta(y)T^\rho_{\mu\nu}^{(+)} + \Theta(-y)T^\rho_{\mu\nu}^{(-)} \end{aligned} \quad (36)$$

This means that the torsion tensor can be written as a distribution and it has no δ -term. This is in contrast to the general relativity where the Riemann tensor includes a δ -term. Consequently the superpotential and the torsion scalar also can be written as distributions

$$S_\rho^{\mu\nu} = \Theta(y)S_\rho^{\mu\nu(+)} + \Theta(-y)S_\rho^{\mu\nu(-)} \quad (37)$$

and

$$T = S_\rho^{\mu\nu}T^\rho_{\mu\nu} = \Theta(y)T^{(+)} + \Theta(-y)T^{(-)} \quad (38)$$

Now we turn our attention to the 5-dimensional teleparallel field equations

$$e^{-1}\partial_\mu(ee^\rho_l S_\rho^{\mu\nu}) - e^\lambda_l T^\rho_{\mu\lambda} S_\rho^{\nu\mu} - \frac{1}{4}e^\nu_l T = 4\pi G e^\rho_l \Xi^\nu_\rho \quad (39)$$

where Ξ^ν_ρ is the 5-dimensional energy-momentum tensor. Any discontinuity (δ -term) in the left hand side of the above equation should be related to the matter energy-momentum tensor on the brane via the second junction condition which we seek to derive here. From equations (36), (37) and (38) and the form of the field equations, it is obvious that the only term which is capable of producing any δ -term and discontinuity in (39) is the first term while the other two terms have no δ -term. Using (37) we'll have

$$\begin{aligned} \partial_\mu(S_\rho^{\mu\nu}) &= \Theta(y)\partial_\mu(S_\rho^{\mu\nu(+)} + \Theta(-y)\partial_\mu(S_\rho^{\mu\nu(-)}) \\ &\quad + \delta(y)[S_\rho^{\mu\nu}]n_\mu \end{aligned} \quad (40)$$

Assuming the bulk to be empty except for a cosmological constant Λ_5 , the 5-dimensional energy-momentum tensor can be decomposed as

$$\Xi^\nu_i = -\Lambda_5 e^\nu_i + \delta(y)\Omega^\nu_i \quad (41)$$

where Ω^ν_i is the matter energy-momentum tensor on the brane. Equalling the δ -terms on the two sides of the teleparallel field equation (39), we get

$$e^\rho_i [S_\rho^{\mu\nu}] n_\mu = 4\pi G \Omega^\nu_i \quad (42)$$

This is the second junction condition. It implies that the jump in the superpotential tensor across the brane is related to the matter energy-momentum tensor confined to the brane.

V. INDUCED TELEPARALLEL FIELD EQUATIONS ON THE BRANE

All the elements required to derive the induced teleparallel field equations on the brane are now prepared. The quantities can be evaluated on either side of the brane by imposing the Z_2 -symmetry. Using the junction condition (42) to relate $S_\rho^{\mu\nu} n_\mu$ terms in (23) to the matter energy-momentum on the brane, we have

$${}^{(4)}F_l^\nu = -\Lambda_5 h_l^\nu + (4\pi G_5)^2 \Pi_l^\nu + E_l^\nu \quad (43)$$

where we have defined

$$\begin{aligned} \Pi_l^\nu &= -\frac{3}{4}h_\rho^i \Omega_i^\nu \Omega_l^\rho + \frac{3}{8}h_l^\rho \Omega \Omega_\rho^\nu \\ &\quad + \frac{1}{32}h_l^\nu \Omega_i^\rho \Omega_\rho^i + \frac{1}{32}h_l^\nu \Omega^2 \\ &\quad + \frac{1}{4}\delta_\rho^\nu \Omega \Omega_l^\rho + \frac{1}{4}\delta_l^\nu \Omega^2 \end{aligned} \quad (44)$$

and

$$\begin{aligned} E_l^\nu &= n^\rho n_l \partial_\mu (S_\rho^{\mu\nu}) + S_\rho^{\mu\nu} (n^\rho \partial_\mu n_l) \\ &\quad + S_\rho^{\mu\nu} (n^l \partial_\mu n_\rho) + h_l^\rho S_\rho^{\mu\nu} (n^i \partial_\mu n_i) \\ &\quad + \left[n^\mu n_\rho n_i n^k e_j^k + n^\nu n_\rho n_j n^k e_i^\mu + n^\mu n_i n_j n^\nu e_\rho^k \right. \\ &\quad \left. - n^k n_\rho e_i^\mu e_j^\nu - n^\mu n_i e_\rho^k e_j^\nu - n^\mu n_\rho n_i n^k n^\nu n_j \right] S_k^{ij} \partial_\mu (h h_l^\rho) \end{aligned} \quad (45)$$

By further decomposing the brane stress-energy tensor into the brane tension and energy-momentum tensor of the matter

$$\Omega_l^\nu = -\lambda h_l^\nu + \tau_l^\nu \quad (46)$$

where λ is the tension of the brane in 5 dimensions and τ_l^ν is the energy-momentum tensor, the equation (43) can be written as

$${}^{(4)}F_l^\nu = -\Lambda_4 h_l^\nu + 4\pi G_4 \tau_l^\nu + (4\pi G_5)^2 \pi_l^\nu + E_l^\nu \quad (47)$$

where

$$\Lambda_4 = \Lambda_5 + (4\pi G_5)^2 \lambda^2 \quad (48)$$

and

$$G_4 = \frac{(4\pi G_5)^2 \lambda}{3\pi} \quad (49)$$

and λ is the brane tension and h_l^ν is the induced vierbein.

We also have

$$\pi_l^\nu = -\frac{3}{4}\tau_\rho^\nu \tau_l^\rho + \frac{3}{8}\tau \tau_l^\nu + \frac{3}{8}h_l^\nu \tau_i^\rho \tau_\rho^i + \frac{3}{16}h_l^\nu \tau^2 + \frac{1}{4}\delta_l^\nu \tau_i^\rho \tau_\rho^i + \frac{1}{4}\delta_l^\nu \tau^2 \quad (50)$$

Equation (47) is the induced teleparallel field equation on the brane. The last two terms in (50) are not present in GR version of field equations. They are extra terms arising from extra degrees of freedom in teleparallel gravity. By conducting the calculation in the Gaussian Normal

coordinates(GNC), the E_l^ν term that carries the effects of the bulk geometry on the brane, will be greatly simplified. In this coordinate system the only non-zero terms of E_l^ν are:

$$E_l^\nu = n_l n^\rho \partial_\mu (S_\rho^{\mu\nu}) + \frac{5}{2} T_\alpha^{\alpha\nu} n_l \quad (51)$$

where $T_\rho^{\mu\nu}$ is the bulk torsion tensor and the normal vector is purely in the direction of the extra dimension (only its 5th component is nonzero). The torsion tensor can be decomposed into three components [17], namely a vector part:

$$V_\mu = T_{\nu\mu}^\nu \quad (52)$$

an axial part:

$$A^\mu = \frac{1}{6} \epsilon^{\mu\nu\rho\sigma} T_{\nu\rho\sigma} \quad (53)$$

and a purely tensorial part:

$$P_{\lambda\mu\nu} = \frac{1}{2} (T_{\lambda\mu\nu} + T_{\mu\lambda\nu}) + \frac{1}{6} (g_{\nu\lambda} V_\mu + g_{\nu\mu} V_\lambda) - \frac{1}{3} g_{\lambda\mu} V_\nu \quad (54)$$

it is obvious from equation (51) that the E_l^ν consists only the vector part of the bulk torsion tensor. It may be useful to remember that in GR the corresponding term is given by the projection of the bulk Weyl tensor into the brane. In the cosmological context, when the Friedmann equation come to account, the corresponding term in GR results in the so called dark radiation term. This term plays a crucial role in studying the dynamics of cosmological perturbations. The role of this term in teleparallel gravity is currently under investigation by the present authors.

VI. CONCLUSION AND DISCUSSION

It is a well known fact that teleparallel theory and its extensions are not locally Lorentz invariant. As a result in four dimensions, there are 10 independent components of the metric but 16 independent component of the tetrad. Also there are infinite number of tetrads that can be chosen to construct any given metric. Each of these tetrads are related through a Lorentz transformation (or rotation) in the tangent space. In this paper using a projection vierbein in the form of (12), we have derived the equivalent of the Gauss-Codacci equations in brane teleparallel gravity. Similar to general relativity, any discontinuity in the geometrical part of the field equation should be related to the brane stress-energy content through junction conditions. In general relativity these junction conditions are given in terms of the jump in the extrinsic curvature across the brane. However in teleparallel gravity the jump in the normal part of the superpotential tensor is related to the matter on the brane (equation (42)). Using these relations, we derived the induced teleparallel field equation on the brane (equation (47)). Compared to general relativity there are two extra

terms in the teleparallel version of the field equation corresponding to extra degrees of freedom in the theory. In general relativity the term carrying the effects of the bulk on the brane is given by the projection of the bulk Weyl tensor on the brane. However in our brane teleparallel gravity model this term is given by equation (51). This means that the corresponding term in teleparallel gravity consists of the projection of the vector part of the bulk torsion tensor on the brane.

Appendix

In this appendix we present the detailed proof of the Gauss equation in teleparallel gravity, equation (18). From the definition of the covariant derivative on the brane, equation (14), we have for any arbitrary vector X_ρ

$$\begin{aligned} D_\mu D_\nu X_\rho &= h_\nu^l h_\rho^m h_\mu^k h_l^\alpha h_m^\beta \nabla_k \left[h_\alpha^i h_\beta^j \nabla_i (e_j^\gamma X_j) \right] \\ &= h_\nu^l h_\rho^m h_\mu^k h_l^\alpha h_m^\beta h_\alpha^i h_\beta^j \nabla_k \nabla_i (e_j^\gamma X_j) \\ &\quad + h_\nu^l h_\rho^m h_\mu^k h_l^\alpha h_m^\beta \nabla_k (h_\alpha^i h_\beta^j) \nabla_i (e_j^\gamma X_j) \end{aligned} \quad (A.1)$$

Swapping μ and ν and subtracting, we have from equation (17)

$$\begin{aligned} {}^{(N-1)}T_{\mu\nu}^\delta D_\delta X_\rho &= h_\nu^l h_\rho^m h_\mu^k h_l^\alpha h_m^\beta h_\alpha^i h_\beta^j \nabla_k \nabla_i (e_j^\gamma X_j) \\ &\quad + h_\nu^l h_\rho^m h_\mu^k h_l^\alpha h_m^\beta \nabla_k (h_\alpha^i h_\beta^j) \nabla_i (e_j^\gamma X_j) \\ &\quad - h_\mu^l h_\rho^m h_\nu^k h_l^\alpha h_m^\beta h_\alpha^i h_\beta^j \nabla_i \nabla_k (e_j^\gamma X_j) \\ &\quad - h_\mu^l h_\rho^m h_\nu^k h_l^\alpha h_m^\beta \nabla_i (h_\alpha^i h_\beta^j) \nabla_k (e_j^\gamma X_j) \end{aligned} \quad (A.2)$$

We now use the definition of the projection vierbein (12) and the fact that the Weitzenböck covariant derivative of the tetrad field is zero. The result after some manipulation is

$$\begin{aligned} {}^{(N-1)}T_{\mu\nu}^\delta D_\delta X_\rho &= h_\nu^l h_\rho^m h_\mu^k h_l^\alpha h_m^\beta e_j^\gamma {}^{(N)}T_{ki}^\alpha \nabla_\alpha X_\gamma \\ &\quad + e_j^\gamma h_\nu^l h_\rho^m h_\mu^k h_l^\alpha h_m^\beta \nabla_k \left[(e_\alpha^i - n^i n_\alpha) (e_\beta^j - n^j n_\beta) \right] \nabla_i X_j \\ &\quad - e_j^\gamma h_\mu^l h_\rho^m h_\nu^k h_l^\alpha h_m^\beta \nabla_i (h_\alpha^i h_\beta^j) \nabla_k X_j \end{aligned} \quad (A.3)$$

We now develop the R.H.S of the above equation

$${}^{(N-1)}T_{\mu\nu}^\delta D_\delta X_\rho = {}^{(N-1)}T_{\mu\nu}^\delta e_i^\gamma h_\delta^j h_\rho^i e_j^\alpha \nabla_\alpha X_\gamma \quad (A.4)$$

Combining the results and noting that $h_\mu^i n^\mu = h_\nu^j n_j = 0$, we'll have

$${}^{(N-1)}T_{\mu\nu}^\delta e_i^\gamma h_\delta^j h_\rho^i e_j^\alpha \nabla_\alpha X_\gamma = h_\nu^l h_\rho^m h_\mu^k h_l^\alpha h_m^\beta e_j^\gamma {}^{(N)}T_{ki}^\alpha \nabla_\alpha X_\gamma \quad (A.5)$$

Removing the arbitrary vector and rearranging, we get the desired result.

$${}^{(N-1)}T_{\mu\nu}^\rho = h_k^\rho h_\nu^j h_\mu^i {}^{(N)}T_{ij}^k \quad (A.6)$$

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